High-Energy Behavior of the Scattering Amplitude for Negative Momentum Transfer*

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The high-energy behavior of the scattering amplitude is investigated in the real negative region of momentum transfers $-t$, for t below the threshold $t=4m^2$ of the crossed channel. If one assumes the existence of bound states in the crossed t channel with angular momenta larger than one, one can show that the high-energy scattering amplitude behaves as if dominated by a Regge trajectory $\alpha(t)$ of even signature and the quantum numbers of the vacuum. It is shown that $\alpha(t)$ is continuous in the open interval $(0,4m^2)$, and an upper bound for $\alpha(t)$ is given under the assumption of analyticity in the domain $\mathrm{Re}t^{1/2}\leq 2m$.

1. INTRODUCTION

It is known that the Froissart bound[1] for the relativistic scattering amplitude $F(s,t)$ can be deduced from analyticity in the Lehmann ellipse plus the weak assumptions that the absorptive part of $F(s,t)$ is analytic in t in the neighborhood of some finite positive interval $(0,t_0)$ and is bounded there by a power of s.[2,3] It has now also been proved,[4] by using, in addition, analyticity in the s plane, that if $F(s,t)$ has no poles in t corresponding to bound states with angular momentum larger than one in the interval $(0,4m^2)$ the dispersion integrals are actually convergent with only two subtractions. We shall discuss here the asymptotic behavior of $F(s,t)$ assuming the existence of poles with angular momentum larger than one. Although no elementary bosons exist with spin higher than one, this analysis has interest in itself as it discloses a connection between the high-energy behavior of the scattering amplitude and the angular momenta of the assumed bound states according to the pattern of a leading Regge trajectory $\alpha(t)$ of even signature and the quantum numbers of the vacuum. It is shown that $\alpha(t)$ is continuous in the open interval $(0,4m^2)$.

We have also obtained an upper bound for $\alpha(t)$, assuming analyticity inside the parabola $\mathrm{Re}\sqrt{t}=2m$. This parabola is the limit as $k^2\to\infty$, of the ellipse of convergence of the Legendre polynomial expansion.

2. BOUND STATES AND HIGH ENERGY BEHAVIOR

Let $F(s,u,t)$ be the scattering amplitude describing three processes:

$$\begin{array}{ll} \mathrm{I} & A+B\to A'+B', \\ \mathrm{II} & A+\bar{B}'\to A'+\bar{B}, \\ \mathrm{III} & A+\bar{A}'\to B'+\bar{B}, \end{array}$$

where A and B are two scalar particles of mass M_A and M_B, respectively. The first two processes are elastic scattering and the last one is a collision in a state with the quantum numbers of the vacuum. The variables s, u, and t are related by

$$s+t+u=2(M_A^2+M_B^2). \tag{1}$$

We assume as in Ref. 4 that $F(s,u,t)$ is an analytic function of t in a certain domain $\mathfrak{D}$ as required to derive the Froissart bound, is bounded by a power s^N of s and, in addition, for fixed t inside $\mathfrak{D}$, it is an analytic function of s with cuts along $s=(M_A+M_B)^2$ to $+\infty$ and $u=(M_A+M_B)^2$ to $+\infty$. The domain $\mathfrak{D}$ includes a neighborhood of the positive real axis from $t=0$ to $t=4m^2$ with the exception of a finite number of points where $F(s,t,u)$ has simple poles. Here m is the mass of the least massive particle, say the pion mass. One can show[4] that given a positive $\epsilon<1$ one can find a real $t_\epsilon>0$ and independent of s such that for $t<t_\epsilon$, $F(s,u,t)$ is bounded by $s^{1+\epsilon}$. Therefore, for fixed $t<t_\epsilon$, one can write a dispersion relation for $F(s,t,u)$ with only two subtractions:

$$F(s,u,t)=C_0(t)+C_1(t)(s-u)+\frac{s^2}{\pi}\int^{\infty}\frac{A_1(s',t)}{(s'-s)s'^2}ds'+\frac{u^2}{\pi}\int^{\infty}\frac{A_2(u',t)}{(u'-u)u'^2}du'. \tag{2}$$

The dispersion integrals may extend below the elastic threshold $s_0(u_0)=(M_A+M_B)^2$ but above this threshold each absorptive amplitude and *all* its derivatives with respect to t are positive definite, for t in the interval $(0,4m^2)$. Now for t in this interval, $F(s,t,u)$ is bounded by s^N so that one can write a dispersion relation with $N+1$ subtractions:

$$F(s,u,t)=C_0(t)+C_1(t)(s-u)+\sum_{n=2}^{N}(I_{1n}(t)s^n+I_{2n}(t)u^n)+\frac{s^{N+1}}{\pi}\int^{\infty}\frac{A_1(s',t)}{(s'-s)s'^{N+1}}ds'+\frac{u^{N+1}}{\pi}\int^{\infty}\frac{A_2(u',t)}{(u'-u)u'^{N+1}}du'. \tag{3}$$

For $t<t_\epsilon$ a comparison of (2) and (3) shows that $C_0(t)$

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[1] M. Froissart, Phys. Rev. 123, 1054 (1961).
[2] A. Martin, Phys. Rev. 129, 1432 (1963).
[3] M. Greenberg and F. E. Low, Phys. Rev. 124, 2047 (1961).
[4] Y. Jin and A. Martin (to be published).

and $C_1(t)$ are the same in the two expressions and

$$I_{1n}(t)=\frac{1}{\pi}\int\frac{A_1(s',t)}{s'^{n+1}}ds', \tag{4}$$

with a similar expression for $I_{2n}(t)$.

Now let us introduce the variable,

$$z=(s-u)/4k_1k_2 \tag{5}$$

where $k_1=\frac{1}{2}(t-4M_A{}^2)^{1/2}$, $k_2=\frac{1}{2}(t-4M_B{}^2)^{1/2}$ are the initial and final momenta in the center-of-mass system for process III and $z=\cos\theta$, where θ is the scattering angle. In the region we are considering both k_1 and k_2 are pure imaginary and the product is real and negative. One can express s and u in terms of z and t by

$$2k_1k_2z=s+k_1{}^2+k_2{}^2=-(u+k_1{}^2+k_2{}^2). \tag{6}$$

Therefore, since $k_1{}^2$ and $k_2{}^2$ are negative in the expansion of s^n or u^n in power series of z all the coefficients of even powers are positive. On the other hand, one can expand z^p in Legendre polynomials of order $l\leq p$ and $(l-p)$ even. Again in this expansion all the coefficients are positive. Therefore, one can finally write

$$\sum_{n=2}^{N}(I_{1n}(t)s^n+I_{2n}(t)u^n)=\sum_{l=0}^{N}C_l(t)P_l(z), \tag{7}$$

where

$$C_l(t)=\sum_{n=l}^{N}\mu_{ln}(t)[I_{2n}(t)+(-1)^lI_{1n}(t)] \tag{8}$$

and for even l, all the μ_{ln} are positive. (Actually the μ_{ln}'s are all positive definite for both even and odd l.) In the real interval $0<t<4m^2$ the only singularities of $F(s,u,t)$ as a function of t are poles corresponding to bound states in the crossed channel III. Let $t_1, t_2, \cdots, t_k$ be the energies of these bound states, $l_1, l_2, \cdots, l_k$ the corresponding angular momenta. In the neighborhood of $t=t_r$ all the coefficients $C_l(t)$ are regular except $C_{l_r}(t)$, which has a pole at $t=t_r$. It is then clear, by the result of Jin and Martin[4] that the representation (2) is valid all through the interval $0\leq t<t_1'$, where t_1' is the first bound state with angular momentum larger than one. Since the residue at this pole behaves like $|z|^{l_1'}$ and, at least in complex directions $|F(s, t_1'-\epsilon)|<c|s|^2$ it follows that $l_1'=2$.

Let us next consider the sequence of bound states with increasing energies $t_1', t_2', \cdots, t_n'$ and *even* angular momenta $l_1', l_2', \cdots, l_n'$ such that l_i' is larger than the angular momenta of all bound states preceding t_i'. Let us suppose that in the interval $0\leq t<t_i'$ the representation (3) is valid with $N=l_i'-1$. Then by a slight generalization of the argument of Ref. 4 one can show that in the interval $0\leq t<t_{i+1}'$ the representation (3) is valid with $N=l_i'+1$. We shall give the main steps in the proof.

For $t<t_i'$, $I_{1,2n}(t)$ is given by (4) when $n\geq l_i'$. Since $A_{1,2}(s',t)$ and all its derivatives with respect to t are positive (for $s'>s_0$) one can expand $A(s',t)$ in power series of t with positive coefficients. It is then allowed to interchange the order of summation and integration in (4).[5] One thus obtains a power-series representation for $I_n(t)$ with positive coefficients. If t' is the radius of convergence of this series then it is also the first singularity of $I_n(t)$ and vice versa and for $t<t'$ the integral representation still holds.[5] Now since the coefficients $\mu_{ln}(t)$ in (8) are all positive analytic functions of t then, for even $l\geq l_i'$, t' is also a singularity of $C_l(t)$. Since by hypotheses, all $C_l(t)$ with even $l>l_i'$ are regular in the interval $0\leq t<t_{i+1}'$, it follows that the representation (4) holds for $n\geq l_i'+2$ and therefore $F(s,u,t)$ may be represented by (3) with $N=l_i'+1$. Thus our assertion is proved. Since this result is true in the interval $0\leq t\leq t_1'$ its validity in general follows by complete induction. Now using the same argument as before one deduces that:

$$l_{i+1}'=l_i'+2. \tag{9}$$

It may happen that in the interval (t_i', t_{i+1}') there exists a bound state t_j with angular momentum $l_j=l_i'+1$. Since for odd l the expression (8) involves the difference of the two functions $I_{1n}(t)$ and $I_{2n}(t)$ it is not in general true that for $t<t_j$ (3) holds with $N=l_j-1$. It is however obvious that, for $t\geq t_j$, at least l_j+1 subtractions are required.

From the above considerations it is clear that if the angular momentum l_j (even or odd) of a bound state t_j is larger than all the preceding ones the angular momentum of the next bound state with the same property is either l_j+1, or l_j+2 if l_j is even.

Another result which emerges from this analysis is that for all the bound states t_i' as previously defined, the residues are negative. In fact as one approaches the pole t_i' from below, $C_{l_i'}(t)$ will be given by (8) and is positive. Therefore the residue is negative.

3. PROPERTIES OF $\alpha(t)$

Let us now define a function $\alpha(t)$ as the limiting value of the set of real numbers α_i for which both integrals

$$I_{1,2\alpha_i}=\int_{s0}^{\infty}\frac{A_{1,2}(s',t)}{s'^{\alpha_i+1}}ds' \tag{10}$$

are convergent.[6] We shall first show that $\alpha(t)$ is continuous in the open interval $(0,4m^2)$. Let us take in the t plane three circles with origin at $t=0$ and increasing radii t, $t+\delta$ and $t_0=4m^2$, respectively. These circles are inside the domain $\mathfrak{D}$ of analyticity in t of $A(s,t)$ and on each circle $|A(s,t)|$ is maximum on the positive real axis. Then applying to $A(s,t)$, Hadamard's three circles theorem,[7] one obtains

$$A(s, t+\delta)<A(s,t)^{\xi_1}A(s,t_0)^{\xi_2}, \tag{11}$$

[5] E. C. Titchmarsh, *The Theory of Functions* (Oxford University Press, New York, 1939), 2nd ed., p. 44.
[6] This definition was suggested by A. Martin.
[7] E. C. Titchmarsh, Ref. 5, p. 172.

where

$$\xi_1=\ln\left(\frac{t_0}{t+\delta}\right)\Big/\ln\left(\frac{t_0}{t}\right);\quad \xi_2=\ln\left(\frac{t+\delta}{t}\right)\Big/\ln\left(\frac{t_0}{t}\right) \tag{12}$$

and $\xi_1+\xi_2=1$. Since we are excluding the points $t=0$ and $t=t_0$ one can take $\delta_0<t<t_0-\delta_0$, where δ_0 is arbitrarily small. Then for $\delta<\delta_0$ one has:

$$\xi_2<\delta[t\ln(t_0/t)]^{-1}<2(\delta/\delta_0). \tag{13}$$

But $A(s,t_0)$ is bounded by $(s/s_0)^N$, therefore (11) gives

$$A(s,t+\delta)<A(s,t)(s/s_0)^{\kappa\delta}, \tag{14}$$

where $\kappa=2N/\delta_0$. Therefore, given an ϵ one can choose a $\delta_1=\epsilon/\kappa$ such that, for $\delta<\min\{\delta_0,\delta_1\}$, one has

$$\int_{s_0}^{\infty}\frac{A(s,t+\delta)}{s^{\alpha(t)+\epsilon+1}}ds<s_0^{-\kappa\delta}\int_{s_0}^{\infty}\frac{A(s,t)}{s^{\alpha(t)+\kappa(\delta_1-\delta)+1}}<\infty. \tag{15}$$

Hence $|\alpha(t+\delta)-\alpha(t)|<\epsilon$ so that $\alpha(t)$ is continuous. If $F(s,u,t)$ has a Regge behavior, $\alpha(t)$ coincides with the Pomeranchuk trajectory. However, even in the general sense as defined above $\alpha(t)$ has the properties of the Pomeranchuk trajectory in the interval $(0,4m^2)$, namely that, in the (l,t) plane the leading poles with even angular momentum and quantum numbers of the vacuum lie on $\alpha(t)$ and all the others lie on or below this curve.

Finally let us assume that $A(s,t)$ is actually bounded by $s^{\alpha(t)+\epsilon}$ for whatever small ϵ, and that $\alpha(t)$ is analytic inside the parabola:

$$\mathrm{Re}\sqrt{t}=\sqrt{t_0}=2m. \tag{16}$$

This parabola is the limit as $s^2\to\infty$, of the ellipse of convergence of the Legendre polynomial expansion. Since in the Legendre polynomial expansion of $A(s,t)$ all the coefficients are positive, for all t on or inside the parabola (10) $\mathrm{Re}\,\alpha(t)$ has an absolute maximum at $t=t_0$. Then for λ real and positive

$$\varphi(t)=\exp\lambda[\alpha(t)-\alpha(t_0)]$$

is bounded by one in the same region. Now the interior of the parabola is analytically mapped into the interior of the unit circle by the transformation[8]

$$z=tg^2\left[\frac{\pi}{4}\left(\frac{t}{t_0}\right)^{1/2}\right]. \tag{17}$$

Therefore, one can apply Pick's inequality[9] to the function $\varphi[t(z)]$. One obtains (for real positive t)

$$e^{\lambda\alpha(t)}<\frac{e^{\lambda\alpha(0)}+ze^{\lambda\alpha(t_0)}}{1+ze^{\lambda[\alpha(0)-\alpha(t_0)]}}. \tag{18}$$

In the limit $\lambda\to 0$, (12) becomes

$$\alpha(t)<\alpha(0)+[2z/(1+z)][\alpha(t_0)-\alpha(0)]$$

or

$$\alpha(t)<\alpha(t_0)-\cos\left[\frac{\pi}{2}\left(\frac{t}{t_0}\right)^{1/2}\right][\alpha(t_0)-\alpha(0)], \tag{19}$$

which is an upper bound for $\alpha(t)$ joining the values at $t=0$ and $t=t_0$. Considering that the absence of $\pi-\pi$ bound states imply $\alpha(t_0)<2$ and since $\alpha(0)\leq 1$, an absolute upper bound for the Pomeranchuk trajectory in the interval $(0,4m^2)$ is

$$\alpha(t)=2-\cos\left[\frac{\pi}{2}\left(\frac{t}{t_0}\right)^{1/2}\right]. \tag{20}$$

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[8] A. R. Forsyth, *Theory of Functions of a Complex Variable* (Cambridge University Press, New York, 1918), 3rd ed., p. 619.

[9] C. Caratheodory, *Theory of Functions of a Complex Variable* (Chelsea Publishing Company, New York, 1960), 2nd ed., Vol. II, p. 14.